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Universal Spin Structure

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Any Dirac spin structure on a world manifold x is a subbundle of the composite spinor bundle $S \to \Sigma_T \to X$, where $\Sigma_T \to X$ is a bundle of tetrad gravitational fields. The bundle S admits general covariant transformations that enable us to discover the energy-momentum conservation law in gauge gravitation theory.

1. INTRODUCTION

Metric and metric-affine theories of gravity are formulated on the natural bundles $Y \to X$ (e.g., tensor bundles) which admit the canonical lifts of diffeomorphisms of the base X. These lifts are general covariant transformations of Y. The invariance of a gravitational Lagrangian under these transformations leads to the energy-momentum conservation law, where the gravitational energy-momentum flow reduces to the generalized Komar superpotential (Novotný, 1984; Borowiec *et al.*, 1994; Giachetta and Sardanashvily, 1996; Giachetta and Mangiarotti, 1997; Sardanashvily, 1997). Difficulties arise in gauge gravitation theory in the presence of Dirac fermion fields. The corresponding spin structure is associated with a certain gravitational field, and it is not preserved under general covariant transformations.

To overcome these difficulties, we will consider the universal twofold covering group $\widetilde{GL}_4$ of the general linear group $GL_4 = GL^+(4, \mathbf{R})$ and the corresponding twofold covering bundle $\widetilde{LX}$ of the bundle of linear frames LX (Dabrowski and Percacci, 1986; Lawson and Michelson, 1989; Switt, 1993). One can consider the spinor representations of the group $\widetilde{GL}_4$, which, however, are infinite dimensional (Hehl *et al.*, 1995). At the same time, the following procedure enables us to not exceed the scope of standard fermion models. The total space of the $\widetilde{GL}_4$-principal bundle $\widetilde{LX} \to X$ is the L_s-principal bundle $\widetilde{LX} \to \Sigma_T$ with the structure group $L_s = SL(2, \mathbf{C})$ over the quotient bundle

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$$\Sigma_T = \widetilde{L}X / L_s \to X$$

whose sections are tetrad gravitational fields h. Let us consider the spinor bundle

$$S = (\widetilde{L}X \times V)/L_s$$

associated with the principal bundle $\widetilde{L}X \to \Sigma_T$. Given a tetrad field h, the restriction of S to $h(X) \subset \Sigma_T$ is a subbundle of the composite spinor bundle

$$S \to \Sigma_T \to X$$

which is exactly the spin structure associated with the gravitational field h. General covariant transformations of the frame bundle LX and, consequently, of the bundles $\widetilde{L}X$ and S are defined.

2. PRELIMINARIES

Manifolds throughout are real, finite-dimensional, Hausdorff, second-countable, and connected. By a world manifold X is meant a 4-dimensional manifold which is noncompact, orientable, and parallelizable in order for a pseudo-Riemannian metric, a spin structure, and a causal space-time structure to exist on X. Note that every noncompact manifold admits a pseudo-Riemannian metric, and a noncompact 4-dimensional manifold X has a spin structure iff it is parallelizable. Moreover, this spin structure is unique (Geroch, 1968; Avis and Isham, 1980).

Let π_{LX}: $LX \to X$ be the principal bundle of oriented linear frames in the tangent spaces to a world manifold X (or simply the frame bundle). Its structure group is GL_4. A world manifold X, by definition, is parallelizable iff the frame bundle $LX \to X$ is trivial. Given the holonomic frames $\{\partial_\mu\}$ in the tangent bundle TX, every element $\{H_a\}$ of the frame bundle LX takes the form $H_a = H_a^\mu \partial_\mu$, where H_a^μ is a matrix element of the group GL_4. The frame bundle LX is provided with the bundle coordinates (x^λ, H_a^μ). In these coordinates, the canonical action of the structure group GL_4 on LX reads

$$R_g: \quad H_a^\mu \mapsto H_b^\mu g_a^b, \qquad g \in GL_4$$

The frame bundle LX is equipped with the canonical $\mathbf{R}^4$-valued 1-form

$$\theta_{LX} = H_\mu^a dx^\mu \otimes t_a \tag{2.1}$$

where $\{t_a\}$ is a fixed basis for $\mathbf{R}^4$ and H_μ^b is the inverse matrix of H_a^μ.

The frame bundle $LX \to X$ belongs to the category of natural bundles. Every diffeomorphism f of X gives rise canonically to the automorphism

$$\tilde{f}: \quad (x^\lambda, H^\lambda_a) \mapsto (f^\lambda(x), \partial_\mu f^\lambda H^\mu_a) \tag{2.2}$$

of LX and to the corresponding automorphisms (general covariant transformations)

$$\tilde{f}: \quad T = (LX \times V)/GL_4 \to (\tilde{f}(LX) \times V)/GL_4$$

of any fiber bundle T associated with LX. In particular, if $T = TX$, the lift $\tilde{f} = Tf$ is the familiar tangent morphism to f.

The lift (2.2) yields the canonical horizontal lift $\tilde{\tau}$ of every vector field τ on X onto the principal bundle LX and the associated bundles. The canonical lift of τ over LX is defined by the relation $\mathbf{L}_{\tilde{\tau}}\, \theta_{LX} = 0$. The corresponding canonical lift of τ onto the tensor bundle

$$(\overset{m}{\otimes} TX) \otimes (\overset{k}{\otimes} T^*X)$$

reads

$$\tilde{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^{\alpha_1} \dot{x}^{\nu\alpha_2\cdots\alpha_m}_{\beta_1\cdots\beta_k} + \cdots - \partial_{\beta_1} \tau^\nu \dot{x}^{\alpha_1\cdots\alpha_m}_{\nu\beta_2\cdots\beta_k} - \cdots] \frac{\partial}{\partial \dot{x}^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k}} \tag{2.3}$$

A pseudo-Riemannian metric g on a world manifold X, called a world metric, is represented by a section of the metric bundle

$$\Sigma_{PR} = GLX/O(1, 3) \tag{2.4}$$

where by GLX is meant the bundle of all linear frames in TX, and $O(1, 3)$ is the complete Lorentz group. Since X is oriented, Σ_{PR} is associated with the bundle LX of oriented frames in TX. Its typical fiber is the quotient space $GL(4, \mathbf{R})/O(1, 3)$, homeomorphic to the topological space $\mathbf{RP}^3 \times \mathbf{R}^7$, where by $\mathbf{RP}^3$ is meant the 3-dimensional real projective space. For the sake of simplicity, we will often identify the metric bundle with an open subbundle of the tensor bundle $\Sigma_{RP} \subset \overset{2}{\vee} TX$ with coordinates $(x^\lambda, \sigma^{\mu\nu})$. By $\sigma_{\mu\nu}$ are meant the components of the inverse matrix, and $\sigma = \det(\sigma_{\mu\nu})$. The canonical lift $\tilde{\tau}$, (2.3), onto Σ_{PR} reads

$$\tilde{\tau} = \tau^\lambda \partial_\lambda + (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha}) \frac{\partial}{\partial \sigma^{\alpha\beta}} \tag{2.5}$$

A linear connection on TX and T^*X, called a world connection, is given by coordinate expressions

$$K = dx^\lambda \otimes \left(\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \frac{\partial}{\partial \dot{x}^\mu} \right) \tag{2.6}$$

$$K^* = dx^\lambda \otimes \left(\partial_\lambda - K_\lambda{}^\mu{}_\nu \dot{x}_\mu \frac{\partial}{\partial \dot{x}_\nu} \right) \tag{2.7}$$

There is one-to-one correspondence between the world connections and the sections of the quotient bundle

$$C_K = J^1 LX/GL_4 \tag{2.8}$$

where by $J^1\ LX$ is meant the first-order jet manifold of the frame bundle $LX \to X$. With respect to the holonomic frames in TX, the bundle C_K is coordinatized by $(x^\lambda, k^\nu_{\lambda\alpha})$, so that, for any section K of $C_K \to X$,

$$k^\nu_{\lambda\alpha} \circ K = K^\nu_{\lambda\alpha}$$

are the coefficients of the world connection K, (2.6). There exists the canonical lift

$$\tilde{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^\alpha k^\nu_{\mu\beta} - \partial_\beta \tau^\nu k^\alpha_{\mu\nu} - \partial_\mu \tau^\nu k^\alpha_{\nu\beta} + \partial_{\mu\beta}\tau^\alpha] \frac{\partial}{\partial k^\alpha_{\mu\beta}} \tag{2.9}$$

onto C_K of a vector field τ on X.

Note that, if a vector field τ is nonvanishing at a point $x \in X$, then there exists a local symmetric connection K around x such that τ is its integral section, i.e., $\partial_\nu \tau^\alpha = K^\alpha_{\nu\beta} \tau^\beta$. Then the canonical lift $\tilde{\tau}$, (2.3), can be found locally as the horizontal lift of τ by K.

3. DIRAC SPINORS

Let M be the Minkowski space equipped with the Minkowski metric which reads

$$\eta = \mathrm{diag}\,(1, -1, -1, -1)$$

with respect to a basis $\{e^a\}$ for M. Let $\mathbf{C}_{1,3}$ be the complex Clifford algebra generated by elements of M. It is isomorphic to the real Clifford algebra $\mathbf{R}_{2,3}$, whose generating space is $\mathbf{R}^5$ with the metric diag$(1, -1, -1, -1, 1)$. Its subalgebra generated by the elements of $M \subset \mathbf{R}^5$ is the real Clifford algebra $\mathbf{R}_{1,3}$.

A spinor space V is defined to be a minimal left ideal of $\mathbf{C}_{1,3}$ (Crawford, 1991; Rodrigues and De Souza, 1993; Obukhov and Solodukhin, 1994). We have the representation

$$\gamma: \quad M \otimes V \to V, \qquad \gamma(e^a) = \gamma^a \tag{3.1}$$

of elements of the Minkowski space $M \subset \mathbf{C}_{1,3}$ by the Dirac γ-matrices on V. Different ideals lead to equivalent representations (3.1). The spinor space V is provided with the spinor metric

$$a(v, v') = \tfrac{1}{2}(v^+\gamma^0 v' + v'^+\gamma^0 v) \tag{3.2}$$

since the element $e^0 \in M$ satisfies the conditions

$$(e^0)^+ = e^0, \qquad (e^0 e)^+ = e^0 e, \qquad \forall e \in M$$

The Clifford group $G_{1,3}$ comprises all invertible elements l_s of the real Clifford algebra $\mathbf{R}_{1,3}$ such that the corresponding inner automorphisms preserve the Minkowski space $M \subset \mathbf{R}_{1,3}$ that is,

$$l_s e l_s^{-1} = l(e), \qquad e \in M \tag{3.3}$$

where $l \in O(1, 3)$ is a Lorentz transformation of M. Thus, we have an epimorphism of the Clifford group $G_{1,3}$ onto the Lorentz group $O(1, 3)$. Since the action (3.3) of the Clifford group on the Minkowski space M is not effective, one usually considers its pin and spin subgroups. The subgroup $Pin(1, 3)$ of $G_{1,3}$ is generated by elements $e \in M$ such that $\eta(e, e) = \pm 1$. The even part of $Pin(1, 3)$ is the spin group $Spin(1, 3)$. Its component of the unity $L_s \simeq SL(2, \mathbf{C})$ is the twofold universal covering group $z_L{:}L_s \to L$ of the proper Lorentz group $L = SO^0(1, 3)$. Recall that L is homeomorphic to $\mathbf{RP}^3 \times \mathbf{R}^3$. The Lorentz group L acts on the Minkowski space M by the generators

$$L^c_{abd} = \eta_{ad}\delta^c_b - \eta_{bd}\delta^c_a \tag{3.4}$$

The Clifford group $G_{1,3}$ acts on the spinor space V by left multiplications

$$G_{1,3} \ni l_s{:} \quad v \mapsto l_s v, \qquad v \in V$$

This action preserves the representation (3.1), i.e.,

$$\gamma(lM \otimes l_s V) = l_s \gamma(M \otimes V) \tag{3.5}$$

The spin group L_r acts on the spinor space V by the generators

$$L_{ab} = \tfrac{1}{4}[\gamma_a, \gamma_b] \tag{3.6}$$

Since $L^+_{ab}\gamma^0 = -\gamma^0 L_{ab}$, this action preserves the spinor metric (3.2).

Let us consider a bundle of Minkowski spaces $MX \to X$ over a world manifold X. This bundle is extended to the bundle of Clifford algebras CX with the fibers $C_x X$ generated by the fibers $M_x X$ of MX (Benn and Tucker, 1987; Rodrigues and Vaz, 1996). The bundle CX has the structure group $\mathrm{Aut}(\mathbf{C}_{1,3})$ of inner automorphisms of the Clifford algebra $\mathbf{C}_{1,3}$. This structure group is reducible to the Lorentz group $SO(1, 3)$, and the bundle of Clifford algebras CX contains the subbundle MX of the generating Minkowski spaces. However, CX does not necessarily contain a spinor subbundle, because a spinor subspace V of $\mathbf{C}_{1,3}$ is not stable under inner automorphisms of $\mathbf{C}_{1,3}$. As has been shown (Benn and Tucker, 1988; Rodrigues and Vaz, 1996), the above-mentioned spinor subbundle S_M exists if the transition functions of CX can be lifted from $\mathrm{Aut}(\mathbf{C}_{1,3})$ to $G_{1,3}$. This agrees with the usual conditions of existence of a spin structure. The bundle MX of Minkowski spaces must be isomorphic to the cotangent bundle T^*X for sections of the spinor bundle

S_M to describe Dirac fermion fields on a world manifold X. In other words, we should consider a spin structure on the cotangent bundle T^*X of X (Lawson and Michelson, 1989). There are several almost equivalent definitions of such a spin structure (Avis and Isham, 1980; Benn and Tucker, 1987; Lawson and Michelson, 1989; Van der Heuvel, 1994). A Dirac spin structure on a world manifold X is said to be a pair (P_s, z_s) of an L_s-principal bundle $P_s \to X$ and a principal bundle morphism

$$z_s: \quad P_s \to LX \tag{3.7}$$

Since the homomorphism $L_s \to GL_4$ factorizes through the epimorphism $L_s \to L$, every bundle morphism (3.7) factorizes through a morphism of P_s onto some L-principal subbundle of the frame bundle LX. It follows that the necessary condition for the existence of a Dirac spin structure on X is that the structure group GL_4 of LX is reducible to the proper Lorentz group L.

4. REDUCED STRUCTURE

Let $\pi_{PX}: P \to X$ be a principal bundle with a structure group G, which acts freely and transitively on P on the right:

$$R_g: \quad p \mapsto pg, \qquad p \in P, \quad g \in G \tag{4.1}$$

Let

$$Y = (P \times V)/G \tag{4.2}$$

be a P-associated bundle with a typical fiber V on which the structure group G acts on the left. By $[p]$ we denote the restriction of the canonical morphism

$$P \times V \to (P \times V)/G$$

to $\{p\} \times V$ and write $[p](v) = (p, v) \cdot G$.

By a principal automorphism of a principal bundle P is meant its automorphism Φ which is equivariant under the canonical action (4.1), that is, $R_g \circ \Phi = \Phi \circ R_g$ for all $g \in G$. A principal automorphism yields the corresponding automorphism

$$\Phi_Y: \quad (P \times V)/G \to (\Phi(P) \times V)/G \tag{4.3}$$

of every P-associated bundle Y, (4.2). An automorphism Φ over Id_X is called vertical.

Let H be a Lie subgroup of G. We have the composite bundle

$$P \to P/H \to X \tag{4.4}$$

where

$$\Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X \tag{4.5}$$

is a P-associated bundle with the typical fiber G/H, and

$$P_\Sigma = P \xrightarrow{\pi_{P\Sigma}} P/H \tag{4.6}$$

is a principal bundle with the structure group H. The structure group G of a principal bundle P is said to be reducible to the subgroup H if there exists an H-principal subbundle P^h of P. This subbundle is called a reduced $G\downarrow H$-structure (Kobayashi, 1972; Gordejuela and Masqué, 1995). Recall the following theorem (Kobayashi and Nomizu, 1963).

Theorem 1. There is one-to-one correspondence $P^h = \pi_{P\Sigma}^{-1}(h(X))$ between the H-principal subbundles P^h of P and the global sections h of the quotient bundle $P/H \to X$. Given such a section h, let us consider the restriction $h^* P_\Sigma$ of the H-principal bundle P_Σ (4.6) to $h(X)$. This is an H-principal bundle over X, which is isomorphic to the reduced subbundle P^h of P.

In general, there are topological obstructions to the reduction of a structure group of a principal bundle to a subgroup. A structure group G of a principal bundle P is reducible to its closed subgroup H if the quotient G/H is homeomorphic to a Euclidean space. In this case, all H-principal subbundles of P are isomorphic to each other as H-principal bundles (Steenrod, 1972). In particular, a structure group is always reducible to its maximal compact subgroup. We have the following assertions (Giachetta *et al.*, 1997).

Proposition 2. Every vertical principal automorphism Φ of the principal bundle $P \to X$ sends a reduced subbundle P^h onto an isomorphic H-principal subbundle $P^{h'}$. Conversely, let two reduced subbundles P^h and $P^{h'}$ of a principal bundle P be isomorphic to each other as H-principal bundles and let Φ: $P^h \to P^{h'}$ be an isomorphism. Then Φ can be extended to a vertical automorphism of P.

Given a reduced subbundle P^h of a principal bundle P, let

$$Y^h = (P^h \times V)/H \tag{4.7}$$

be the P^h-associated bundle with a typical fiber V. If $P^{h'}$ is another reduced subbundle of P which is isomorphic to P^h, the fiber bundles Y^h and $Y^{h'}$ are isomorphic, but not canonically isomorphic in general.

Proposition 3. Let P^h be an H-principal subbundle of a G-principal bundle P. Let Y be the P^h-associated bundle (4.7) with a typical fiber V. If V carries a representation of the whole group G, the fiber bundle Y^h is canonically isomorphic to the P-associated fiber bundle (4.2).

It follows that, given an H-principal subbundle P^h of P, any P-associated bundle Y with the structure group G is canonically equipped with a structure of the P^h-associated fiber bundle Y^h with the structure group H. Briefly, we will write

$$Y = (P \times V)/G = (P^h \times V)/H$$

However, P^h and $P^{h'}$-associated bundle structures on Y are not equivalent because, given bundle atlases Ψ^h of P^h and $\Psi^{h'}$ of $P^{h'}$, the union of the associated atlases of Y has necessarily G-valued transition functions between the charts from Ψ^h and $\Psi^{h'}$.

5. DIRAC SPIN STRUCTURE

Since a world manifold is parallelizable, the structure group GL_4 of the frame bundle LX is reducible to the Lorentz group L. The corresponding L-principal subbundle L^hX is said to be a Lorentz structure.

In accordance with Theorem 1, there is one-to-one correspondence between the L-principal subbundles L^hX of LX and the global sections h of the quotient bundle

$$\Sigma_T = LX/L \tag{5.1}$$

called the tetrad bundle. This is an LX-associated bundle with the typical fiber GL_4/L. Since the group GL_4 is homotopic to its maximal compact subgroup $SO(4)$ and the Lorentz group L is homotopic to $SO(3)$, GL_4/L is homotopic to $SO(4)/SO(3) = S^3$, and homeomorphic to $S^3 \times \mathbf{R}^7$. The bundle (5.1) is the twofold covering of the metric bundle Σ_{PR} (2.4). Its global sections are called the tetrad fields.

Since X is parallelizable, any two Lorentz subbundles L^hX and $L^{h'}X$ are isomorphic to each other. By virtue of Proposition 2, there exists a vertical bundle automorphism Φ of LX which sends L^hX onto $L^{h'}X$. The associated vertical automorphism Φ_Σ of the fiber bundle $\Sigma_T \to X$ transforms the tetrad field h into the tetrad field h'.

Every tetrad field h defines an associated Lorentz atlas $\Psi^h = \{U_\zeta, z_\zeta^h\}$ of LX such that the corresponding local sections z_ζ^h of the frame bundle LX take their values into the Lorentz subbundle L^hX. Given a Lorentz atlas Ψ^h, the pullback

$$z_\zeta^h{}^*\theta_{LX} = h^a \otimes t_a = h_\lambda^a dx^\lambda \otimes t_a \tag{5.2}$$

of the canonical form θ_{LX}, (2.1), by a local section z_ζ^h is said to be a (local) tetrad form. It determines the tetrad coframes

$$h^a(x) = h_\mu^a(x)dx^\mu, \qquad x \in U_\zeta$$

in the cotangent bundle T^*X, which are associated with the Lorentz atlas

Ψ^h. The coefficients h^a_μ of the tetrad forms and the inverse matrix elements $h^\mu_a = H^\mu_a \circ z^h_\zeta$ are called tetrad functions. Given a Lorentz atlas Ψ^h, the tetrad field h can be represented by the family of tetrad functions $\{h^\mu_a\}$. We have the familiar relation $g = h^a \otimes h^b \eta_{ab}$ between tetrad and metric fields.

Given a tetrad field h, let L^hX be the corresponding Lorentz subbundle. Since X is noncompact and parallelizable, the principal bundle L^hX can be extended uniquely to an L_s-principal bundle $P^h \to X$, called the h-associated principal spinor bundle (Geroch, 1968). We have the principal bundle morphism

$$z_h: \quad P^h \to L^hX, \qquad z_h \circ R_g = R_{z_L(g)}, \qquad \forall g \in L_s \tag{5.3}$$

This is the h-associated Dirac spin structure on a world manifold.

Let us consider the L^hX-associated bundle of Minkowski spaces

$$M^hX = (L^hX \times M)/L = (P^h \times M)/L_s \tag{5.4}$$

and the P^h-associated spinor bundle

$$S^h = (P^h \times V)/L_s \tag{5.5}$$

called the h-associated spinor bundle. By virtue of Proposition 3, the bundle M^hX, (5.4), is isomorphic to the cotangent bundle

$$T^*X = (L^hX \times M)/L \tag{5.6}$$

Then there exists the representation

$$\gamma_h: \quad T^*X \otimes S^h = (P^h \times (M \otimes V))/L_s \to (P^h \times \gamma(M \otimes V))/L_s = S^h \tag{5.7}$$

of covectors to X by the Dirac γ-matrices on elements of the spinor bundle S^h. Relative to an atlas $\{z_\zeta\}$ of P^h and to the associated Lorentz atlas $\{z^h_\zeta = z_h \circ z_\zeta\}$ of LX, the representation (5.7) reads

$$y^A(\gamma_h(h^a(x) \otimes v)) = \gamma^{aA}_B y^B(v), \qquad v \in S^h_x$$

where y^A are the corresponding bundle coordinates of S^h, and h^a are the tetrad coframes (5.2). For brevity, we will write

$$\hat{h}^a = \gamma_h(h^a) = \gamma^a, \qquad \hat{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a$$

Sections s_h of the h-associated spinor bundle S^h, (5.5), describe Dirac fermion fields in the presence of the tetrad field h. Indeed, let A_h be a principal connection on S^h, and let

$$D: \quad J^1S^h \to T^*X \underset{S^h}{\otimes} S^h$$

$$D = (y^A_\lambda - A^{ab}_\lambda L^A_{abB} y^B) dx^\lambda \otimes \partial_A$$

be the corresponding covariant differential. The first-order differential Dirac operator is defined on S^h by the composition

$$\Delta_h = \gamma_h \circ D: \quad J^1S^h \to T^*X \otimes S^h \to S^h \tag{5.8}$$

$$y^A \circ \Delta_h = h_a^\gamma \, \gamma_B^{aA}(y_\lambda^B - \tfrac{1}{2} A_\lambda^{ab} \, L_{abB}^A y^B)$$

The h-associated spinor bundle S^h is equipped with the fiber spinor metric

$$a_h(v, v') = \tfrac{1}{2}(v^+\gamma^0 v' + v'^+\gamma^0 v), \qquad v, v' \in S^h$$

Using this metric and the Dirac operator (5.8), one can define the Lagrangian density

$$\begin{aligned} L_h = {} & \{\tfrac{i}{2} h_q^\lambda [y_A^+(\gamma^0\gamma^q)_B^A(y_\lambda^B - \tfrac{1}{2} A_\lambda^{ab} L_{abC}^B y^C) \\ & - (y_{\lambda A}^+ - \tfrac{1}{2} A_\lambda^{ab} y_C^+ L_{ab}^+)(\gamma^0\gamma^q)_B^A y^B] - m y_A^+(\gamma^0)_B^A y^B\} \det(h_\mu^a) \end{aligned} \tag{5.9}$$

on J^1S^h which describes Dirac fermion fields in the presence of a background tetrad field h and a background connection A_h on S^h.

6. SPIN CONNECTIONS

Let us recall the following theorem (Kobayashi and Nomizu, 1963).

Theorem 4. Let $P' \to X$ and $P \to X$ be principle bundles with the structure groups G' and G, respectively. Let Φ: $P' \to P$ be a bundle morphism over X with the corresponding homomorphism $G' \to G$. For every principal connection A' on P', there exists a unique principal connection A on P such that $T\Phi$ sends the horizontal subspaces of A' onto the horizontal subspaces of A.

It follows that every principal (spin) connection

$$A_h = dx^\lambda \otimes (\partial_\lambda + \tfrac{1}{2} A_\gamma^{ab} e_{ab}) \tag{6.1}$$

on P^h defines a principal (Lorentz) connection on L^hX which is given by the same expression (6.1). Conversely, the pullback $z_h^* \omega_A$ on P^h of a connection form ω_A of a Lorentz connection A_h on L^hX is equivariant under the action of group L_s on P^h and, consequently, it is a connection form of a spin connection on P^h. In particular, the Levi-Civita connection of a tetrad field h gives rise to a spin connection

$$A_\lambda^{ab} = \eta^{kb} h_\mu^a (\partial_\lambda h_k^\mu - h_k^\nu \{^\mu{}_{\lambda\nu}\}) \tag{6.2}$$

on the h-associated spinor bundle S^h.

Moreover, every world connection K on a world manifold X also defines a spin connection on an h-associated principal spinor bundle P^h (Giachetta and Mangiarotti, 1997; Sardanashvily, 1997).

In accordance with Theorem 4, every Lorentz connection A_h,(6.1), on a Lorentz subbundle L^hX of LX gives rise to a world connection

$$K_\lambda{}^\mu{}_\nu = h^k_\nu \partial_\lambda h^\mu_k + \eta_{ka} h^\mu_b h^k_\nu A_\lambda^{ab} \tag{6.3}$$

on LX. At the same time, every principal connection K on the frame bundle LX defines a Lorentz principal connection K_h on an L-principal subbundle L^hX as follows.

It is readily observed that the Lie algebra of the general linear group GL_4 is the direct sum

$$\mathbf{g}(GL_4) = \mathbf{g}(L) \oplus \mathbf{m}$$

of the Lie algebra $\mathbf{g}(L)$ of the Lorentz group and a subspace $\mathbf{m} \subset \mathbf{g}(GL_4)$ such that $ad(l)(\mathbf{m}) \subset \mathbf{m}$, for all $l \in L$. Let ω_K be a connection form of a world connection K on LX. Then, by a well-known theorem (Kobayashi and Nomizu, 1963), the pullback over L^hX of the $\mathbf{g}(L)$-valued component ω_L of ω_K is a connection form of a principal connection K_h on the Lorentz subbundle L^hX. To obtain K_h, let us consider a local connection 1-form of the connection K with respect to a Lorentz atlas Ψ^h of LX given by the tetrad forms h^a. This reads

$$z_h^* \omega_K = K_\lambda{}^b{}_k dx^\lambda \otimes e^k_b$$
$$K_\lambda{}^b{}_k = -h^b_\mu \partial_\lambda h^\mu_k + K_\lambda{}^\mu{}_\nu h^b_\mu h^\nu_k$$

where $\{e^k_b\}$ is the basis for the Lie algebra of the group GL_4. Then, the Lorentz part of this form is precisely the local connection 1-form of the connection K_h on L^hX. We have

$$z_h^* \omega_L = \tfrac{1}{2} A_\lambda^{ab} dx^\lambda \otimes e_{ab} \tag{6.4}$$
$$A_\lambda^{ab} = \tfrac{1}{2}(\eta^{kb} h^a_\mu - \eta^{ka} h^b_\mu)(\partial_\lambda h^\mu_k - h^\nu_k K_\lambda{}^\mu{}_\nu)$$

If K is a Lorentz connection A_h, then obviously $K_h = A_h$.

The connection K_h,(6.4), on L^hX yields the corresponding spin connection on S^h

$$K_h = dx^\lambda \otimes [\partial_\lambda + \tfrac{1}{4}(\eta^{kb} h^a_\mu - \eta^{ka} h^b_\mu)(\partial_\lambda h^\mu_k - h^\nu_k K_\lambda{}^\mu{}_\nu) L^A_{abB} y^B \partial A] \tag{6.5}$$

where L_{ab} are the generators (3.6) (Giachetta and Mangiarotti, 1997; Sardanashvily, 1997). Such a connection has been considered by Ponomarev and Obukhov (1982), Aringazin and Mikhailov (1991), and Tucker and Wang (1995). Substituting the spin connection (6.5) into the Dirac operator (5.8) and the Dirac Lagrangian density (5.9), we obtain a description of Dirac fermion fields in the presence of an arbitrary world connection, not only of the Lorentz type.

One can use the connection (6.5) to obtain a horizontal lift onto S^h of a vector field τ on X. This lift reads

$$\tau_{K_h} = \tau^\lambda \partial_\lambda + \tfrac{1}{4}\tau^\lambda(\eta^{kb}h^a_\mu - \eta^{ka}h^b_\mu)(\partial_\lambda h^\mu_k - h^\nu_k K^\mu_{\lambda\nu})L^A_{abB}y^B\partial_A \tag{6.6}$$

Moreover, we have the canonical horizontal lift

$$\tilde{\tau} = \tau^\lambda \partial_\lambda + \tfrac{1}{4}(\eta^{kb}h^a_\mu - \eta^{ka}h^b_\mu)(\tau^\lambda\partial_\lambda h^\mu_k - h^\nu_k\partial_\nu\tau^\mu)L^A_{abB}y^B\partial_A \tag{6.7}$$

of vector fields τ on X onto the h-associated spinor bundle S^h (Sardanashvily, 1997). To construct the canonical lift (6.7), one can write the canonical lift of τ on the frame bundle LX with respect to a Lorentz atlas Ψ^h and take its Lorentz part. Another approach is the following. Let us consider a local nonvanishing vector field τ and a local world symmetric connection K for which τ is an integral section. The horizontal lift (6.6) of τ by means of this connection is given by the expression (6.7). In a straightforward manner, one can check that (6.7) is a well-behaved lift of any vector field τ on X. The canonical lift (6.7) is brought into the form

$$\tilde{\tau} = \tau_{\{\}} - \tfrac{1}{4}(\eta^{kb}h^a_\mu - \eta^{ka}h^b_\mu)h^\nu_k\nabla_\nu\tau^\mu L^A_{abB}y^B\partial_A$$

where $\tau_{\{\}}$ is the horizontal lift (6.6) of τ by means of the spin Levi-Civita connection (6.2) of the tetrad field h, and $\nabla_\nu\tau^\mu$ are the covariant derivatives of τ relative to the same Levi-Civita connection (Kosmann, 1972; Fatibene *et al.*, 1996).

The canonical lift (6.7) fails to be a generator of general covariant transformations because it does not involve transformations of tetrad fields. To define general covariant transformations of spinor bundles, we should consider spinor structures associated with different tetrad fields. The difficulty arises because, though the principal spinor bundles P^h and $P^{h'}$ are isomorphic, the associated structures of bundles of Minkowski spaces M^hX and $M^{h'}X$, (5.4), on the cotangent bundle T^*X are not equivalent, and so are the representations γ_h and $\gamma_{h'}$, (5.7), for different tetrad fields h and h' (Sardanashvily and Zakharov, 1992; Sardanashvily, 1995). Indeed, let

$$t^* = t_\mu dx^\mu = t_a h^a = t'_a h'^a$$

be an element of T^*X. Its representations γ_h and $\gamma_{h'}$, (5.7), read

$$\gamma_h(t^*) = t_a\gamma^a = t_\mu h^\mu_a\gamma^a, \qquad \gamma_{h'}(t^*) = t'_a\gamma^a = t_\mu h'^\mu_a\gamma^a$$

These representations are not equivalent because no isomorphism Φ_s of S^h onto $S^{h'}$ can obey the condition

$$\gamma_{h'}(t^*) = \Phi_s\gamma_h(t^*)\Phi_s^{-1} \qquad \forall t^* \in T^*X$$

It follows that every Dirac fermion field must be described by a pair with a certain tetrad field. We thus observe the phenomenon of symmetry breaking in gauge gravitation theory which exhibits the physical nature of gravity as a Higgs field (Sardanashvily, 1991; Sardanashvily and Zakharov, 1992).

7. SPONTANEOUS SYMMETRY BREAKING

Spontaneous symmetry breaking is a quantum phenomenon modeled by classical Higgs fields. In gauge theory on a principal bundle $P \to X$, the necessary condition for spontaneous symmetry breaking to take place is the reduction of the structure group G of P to the subgroup H of exact symmetries. Higgs fields are described by global sections h of the quotient bundle Σ, (4.5).

In accordance with Theorem 1, the set of Higgs fields h is in bijective correspondence with that of reduced H-principal subbundles P^h of P. Given such a subbundle P^h, let Y^h, (4.7), be the associated bundle with a typical fiber V which admits a representation of the group H of exact symmetries, but not the whole symmetry group G. Its sections s_h describe matter fields in the presence of the Higgs fields h. In general, Y^h is not associated or canonically associated with other H-principal subbundles of P. It follows that V-valued matter fields can be represented only by pairs with Higgs fields. The goal is to describe the totality of these pairs (s_h, h) for all Higgs fields (Sardanashvily, 1992, 1993).

Let us consider the composite bundle (4.4) and the composite bundle

$$Y \xrightarrow{\pi_{Y\Sigma}} \Sigma \xrightarrow{\pi_{\Sigma X}} X \tag{7.1}$$

where $Y \to \Sigma$ is the bundle

$$Y = (P \times V)/H \tag{7.2}$$

associated with the H-principal bundle P_Σ, (4.6). There is the canonical isomorphism i_h: $Y^h \to h^*Y$ of the P^h-associated bundle Y^h to the subbundle of $Y \to X$ which is the restriction $h^*Y = (h^*P \times V)/H$ of the bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$, i.e.,

$$i_h(Y^h) \cong \pi_{Y\Sigma}^{-1}(h(X)) \tag{7.3}$$

Then every global section s_h of Y^h corresponds to the global section $i_h \circ s_h$ of the composite bundle (7.1). Conversely, every global section s of the composite bundle (7.1) which projects onto a section $h = \pi_{Y\Sigma} \circ s$ of the bundle $\Sigma \to X$ takes its values into the subbundle $i_h(Y^h) \subset Y$ in accordance with the relation (7.3). Hence, there is one-to-one correspondence between the sections of the bundle Y^h and the sections of the composite bundle (7.1) which cover h.

Thus, it is precisely the composite bundle (7.1) whose sections describe the above-mentioned totality of the pairs (s_h, h) of matter fields and Higgs fields in gauge theory with broken symmetries (Sardanashvily and Zakharov, 1992; Sardanashvily, 1991, 1995).

The feature of the dynamics of field systems on composite bundles consists in the following. Let the composite bundle Y, (7.1), be coordinatized by $(x^\lambda, \sigma^m, y^i)$, where (x^λ, σ^m) are bundle coordinates on $\Sigma \to X$. Let

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \tag{7.4}$$

be a principal connection on the bundle $Y \to \Sigma$. This connection defines the splitting

$$VY = VY_\Sigma \underset{Y}{\oplus} (Y \underset{\Sigma}{\times} V\Sigma)$$

$$\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A^i_m \dot{\sigma}^m)\partial_i + \dot{\sigma}^m(\partial_m + A^i_m \partial_i)$$

Using this splitting, one can construct the first-order differential operator

$$\tilde{D}: \quad J^1Y \to T^*X \underset{Y}{\otimes} VY_\Sigma$$

$$\tilde{D} = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda - A^i_m \sigma^m_\lambda)\partial_i \tag{7.5}$$

on the composite bundle Y. The operator (7.5) posesses the following property. Given a global section h of Σ, its restriction

$$\tilde{D}_h = \tilde{D} \circ J^1 i_h: \quad J^1Y^h \to T^*X \otimes VY^h \tag{7.6}$$

$$\tilde{D}_h = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda - A^i_m \partial_\lambda h^m)\partial_i$$

to Y^h is exactly the familiar covariant differential relative to the principal connection

$$A_h = dx^\lambda \otimes [\partial_\lambda + (A^i_m \partial_\lambda h^m + A^i_\lambda)\partial_i]$$

on the bundle $Y^h \to X$, which is induced by the principal connection (7.4) on the fiber bundle $Y \to \Sigma$ by the imbedding i_h (Kobayashi and Nomizu, 1963).

8. UNIVERSAL SPIN STRUCTURE

All spin structures on a manifold X which are related to the twofold universal covering groups possess the following properties (Greub and Petry, 1978). Let $P \to X$ be a principal bundle with a structure group G with the fundamental group $\pi_1(G) = \mathbf{Z}_2$. Let $\tilde{G}$ be the universal covering group of G. The topological obstruction to the existence of a $\tilde{G}$-principal bundle $\tilde{P} \to X$ covering the bundle P is given by the Čech cohomology group $H^2(X; \mathbf{Z}_2)$ of X with coefficients in $\mathbf{Z}_2$. Roughly speaking, the principal bundle P defines an element of $H^2(X; \mathbf{Z}_2)$ which must be zero, so that $P \to X$ gives rise to $\tilde{P} \to X$. Nonequivalent lifts of a G-principal bundle P to a $\tilde{G}$-principal bundle are classified by elements of the Čech cohomology group $H^1(X; \mathbf{Z}_2)$.

In particular, the well-known topological obstruction to the existence of a spin structure is the nonzero second Stiefel–Whitney class of X. In the case of 4-dimensional noncompact manifolds, all pseudo-Riemannian spin structures are equivalent.

Let us turn to fermion fields in gauge gravitation theory, basing our consideration on the following two facts (Giachetta *et al.*, 1997).

Proposition 5. The L-principal bundle.

$$P_L = GL_4 \to GL_4/L \tag{8.1}$$

is trivial.

Proposition 6. Since the first homotopy group of the group GL_4 is $\mathbf{Z}_2$, GL_4 admits the universal twofold covering group $\widetilde{GL}_4$. We have the commutative diagram

$$\begin{array}{ccc} \widetilde{GL}_4 & \to & GL_4 \\ \uparrow & & \uparrow \\ L_s & \xrightarrow{z_L} & L \end{array} \tag{8.2}$$

A universal spin structure on X is defined to be a pair consisting of a $\widetilde{GL}_4$-principal bundle $\widetilde{LX} \to X$ and a principal bundle morphism over X

$$\tilde{z}: \quad \widetilde{LX} \to LX \tag{8.3}$$

(Dabrowski and Percacci, 1986; Percacci, 1986; Switt, 1993). This is unique since X is parallelizable. In virtue of Proposition 6, the diagram

$$\begin{array}{ccc} \widetilde{LX} & \xrightarrow{\tilde{z}} & LX \\ \uparrow & & \uparrow \\ P^h & \xrightarrow{z_h} & L^hX \end{array} \tag{8.4}$$

commutes for any tetrad field h (Fulp *et al.*, 1994; Giachetta *et al.*, 1997). It follows that the quotient $\widetilde{LX}/L_s$ is exactly the quotient Σ_T, (5.1), so that there is the commutative diagram

$$\begin{array}{ccc} \widetilde{LX} & \xrightarrow{\tilde{z}} & LX \\ & \searrow \quad \swarrow & \\ & \Sigma_T & \end{array} \tag{8.5}$$

Let us consider the composite bundle

$$\widetilde{LX} \to \Sigma_T \to X \tag{8.6}$$

where $\widetilde{LX} \to \Sigma_T$ is the L_s-principal bundle. For each tetrad field h: $X \to \Sigma_T$, the restriction of the L_s-principal bundle $\widetilde{LX} \to \Sigma_T$ to $h(X) \subset \Sigma_T$ is isomorphic to the h-associated principal spinor bundle P^h. Therefore, the diagram (8.5) is called the universal Dirac spin structure.

The universal Dirac spin structure (8.5) can be regarded as the L_s-spin structure on the bundle of Minkowski spaces

$$E_M = (LX \times M)/L \to \Sigma_T$$

associated with the L-principal bundle $LX \to \Sigma_T$. Since the principal bundles LX and P_L (8.1) are trivial, so is the bundle $E_M \to \Sigma_T$. Hence, it is isomorphic to the pullback

$$\Sigma_T \underset{X}{\times} T^*X \tag{8.7}$$

One can show that a spin structure on this bundle is unique (Giachetta *et al.*, 1997).

Let us consider the composite spinor bundle

$$S \xrightarrow{\pi_{S\Sigma}} \Sigma_T \xrightarrow{\pi_{\Sigma X}} X \tag{8.8}$$

where $S = (\widetilde{LX} \times V)/L_s$ is the spinor bundle $S \to \Sigma_T$ associated with the L_s-principal bundle $\widetilde{LX} \to \Sigma_T$. Given a tetrad field h, there is the canonical isomorphism

$$i_h\colon \quad S^h = (P^h \times V)/L_s \to (h^*\widetilde{LX} \times V)/L_s$$

of the h-associated spinor bundle S_h, (5.5), onto the restriction h^*S of the spinor bundle $S \to \Sigma_T$ to $h(X) \subset \Sigma_T$. Thence, every global section s_h of the spinor bundle S^h corresponds to the global section $i_h \circ s_h$ of the composite spinor bundle (8.8). Conversely, every global section s of the composite spinor bundle (8.8), which projects onto a tetrad field h, takes its values into the subbundle $i_h(S^h) \subset S$.

Let the frame bundle $LX \to X$ be provided with a holonomic atlas, and let the principal bundles $\widetilde{LX} \to \Sigma_T$ and $LX \to \Sigma_T$ have the associated atlases

$\{U_\varepsilon, z_\varepsilon^S\}$ and $\{U_\varepsilon z_\varepsilon = \tilde{z} \circ z_\varepsilon^S\}$. With these atlases, the composite spinor bundle S is equipped with the bundle coordinates $(x^\lambda, \sigma_a^\mu, y^A)$, where $(x^\lambda, \sigma_a^\mu)$ are coordinates on Σ_T such that σ_a^μ are the matrix components of the group element $(T\phi_\zeta \circ z_\varepsilon)(\sigma)$, $\sigma \in U_\varepsilon$, $\pi_{\Sigma X}(\sigma) \in U_\zeta$. For each section h of Σ_T, we have $(\sigma_a^\lambda \circ h)(x) = h_a^\lambda(x)$.

The composite spinor bundle S is equipped with the fiber spinor metric

$$a_S(v, v') = \tfrac{1}{2}(v^+\gamma^0 v' + v'^+\gamma^0 v), \quad \pi_{S\Sigma}(v) = \pi_{S\Sigma}(v')$$

Since the bundle of Minkowski spaces $E_M \to \Sigma_T$ is isomorphic to the pullback bundle (8.7), there exists the representation

$$\gamma\Sigma\colon \quad T^*X \underset{\Sigma_T}{\otimes} S = (\widetilde{LX} \times (M \otimes V))/L_s \to (\widetilde{LX} \times \gamma(M \otimes V))/L_s = S \tag{8.9}$$

given by the coordinate expression

$$\hat{dx}^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma_a^\lambda \gamma^a$$

Restricted to $h(X) \subset \Sigma_T$, this representation recovers the morphism γ_h, (5.7).

Using this representation, one can construct the total Dirac operator on the composite spinor bundle S as follows. Since the bundles which make up the composite bundle (8.6) are trivial, let us consider a principal connection A_Σ, (7.4), on the L_s-principal bundle $\widetilde{LX} \to \Sigma_T$ given by the local connection form

$$A_\Sigma = (A_\lambda^{ab} dx^\lambda + A_\mu^{kab} d\sigma_k^\mu) \otimes L_{ab}, \tag{8.10}$$

where

$$\begin{aligned} A_\lambda^{ab} &= -\tfrac{1}{2}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b)\sigma_k^\nu K_\lambda{}^\mu{}_\nu \\ A_\mu^{kab} &= \tfrac{1}{2}(\eta^{kb}\sigma_\mu^a - \eta^{ka}\sigma_\mu^b) \end{aligned} \tag{8.11}$$

and K is a world connection on X. This connection defines the associated spin connection

$$A_S = dx^\lambda \otimes (\partial_\lambda + \tfrac{1}{2} A_\lambda^{ab} L_{abB}^A y^B \partial_A) + d\sigma_k^\mu \otimes (\partial_\mu^k + \tfrac{1}{2} A_\mu^{kab} L_{abB}^A y^B \partial_A) \tag{8.12}$$

on the spinor bundle $S \to \Sigma_T$. Let h be a global section of $\Sigma_T \to X$ and S^h the restriction of the bundle $S \to \Sigma_T$ to $h(X)$. It is readily observed that the restriction of the spin connection (8.12) to S^h is exactly the spin connection (6.5).

The connection (8.12) yields the first-order differential operator $\tilde{D}$, (7.5), on the composite spinor bundle $S \to X$, which reads

$$\tilde{D}: \quad J^1S \to T^*X \underset{\Sigma_T}{\otimes} S$$

$$\tilde{D} = dx^\lambda \otimes [y^A_\lambda - \tfrac{1}{2}(A^{ab}_\lambda + A^{kab}_\mu \sigma^\mu_{\lambda k}) L^A_{abB} y^B \partial_A \tag{8.13}$$
$$= dx^\lambda \otimes [y^A_\lambda - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k K^\mu_{\lambda\nu}) L^A_{abB} y^B]\partial_A$$

The corresponding restriction $\tilde{D}_h$, (7.6), of the operator $\tilde{D}$, (8.13), to $J^1S^b \subset J^1S$ recovers the familiar covariant differential on the h-associated spinor bundle $S^h \to X$ relative to the spin connection (6.7).

Combining (8.9) and (8.13), we obtain the first-order differential operator

$$\Delta = \gamma_\Sigma \circ \tilde{D}: \quad J^1S \to T^*X \underset{\Sigma T}{\otimes} S \to S \tag{8.14}$$

$$y^B \circ \Delta = \sigma^\lambda_a \gamma^{aB}_A [y^A_\lambda - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k K^\mu_{\lambda\nu}) L^A_{abB} y^B]$$

on the composite spinor bundle $S \to X$. One can think of Δ as being the total Dirac operator on S since, for every tetrad field h, the restriction of Δ to $J^1S^h \subset J^1S$ is exactly the Dirac operator Δ_h,(5.8), on the h-associated spinor bundle S^h in the presence of the background tetrad field h and the spin connection (6.5).

It follows that gauge gravitation theory is reduced to the model of metric-affine gravity and Dirac fermion fields. The total configuration space of this model is the jet manifold J^1Y of the bundle product

$$Y = C_K \underset{\Sigma_T}{\times} S \tag{8.15}$$

where C_K is the bundle of world connections (2.8). It is coordinatized by $(x^\mu, \sigma^\mu_a, k^\alpha_{\mu\beta}, y^A)$.

Let $J^1_\Sigma Y$ denotes the first-order jet manifold of the bundle $Y \to \Sigma_T$. This bundle can be provided with the spin connection

$$A_Y: \quad \to J^1_\Sigma Y \xrightarrow{\mathrm{pr}_2} J^1_\Sigma S$$

$$A_Y = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^{ab}_\lambda L^A_{abB} y^B \partial^A) + d\sigma^\mu_k \otimes (\partial^k_\mu + A^{kab}_\mu L^A_{abB} y^B \partial_A) \tag{8.16}$$

where A^{kab}_μ is given by the expression (8.11), and

$$\tilde{A}^{ab}_\lambda = -\tfrac{1}{2}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\sigma^\nu_k K^\mu_{\lambda\nu}$$

Using the connection (8.16), we obtain the first-order differential operator

$$\tilde{D}_Y: \quad J^1Y \to T^*X \underset{\Sigma_T}{\otimes} S$$

$$\tilde{D}_Y = dx^\lambda \otimes [y^A_\lambda - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k k^\mu_{\lambda\nu}) L^A_{abB} y^B]\partial_A \tag{8.17}$$

and the total Dirac operator

$$\Delta_Y = \gamma_\Sigma \circ \tilde{D}: \quad J^1Y \to T^*X \underset{\Sigma_T}{\otimes} S \to S \tag{8.18}$$

$$y^B \circ \Delta_Y = \sigma^\lambda_a \gamma^{aB}{}_A [y^A_\lambda - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k k^\mu{}_{\lambda\nu}) L^A{}_{abB} y^B]$$

on the bundle $Y \to X$.

Given a section $K: X \to C_K$, the restrictions of the spin connection A_Y, (8.16), the operator $\tilde{D}_Y$, (8.17), and the Dirac operator Δ_Y, (8.18), to K^*Y are exactly the spin connection (8.12) and the operators (8.13) and (8.14), respectively.

The total Lagrangian density on the configuration space J^1Y of the metric-affine gravity and fermion fields is the sum $L = L_{MA} + L_D$ of the metric-affine Lagrangian density L_{MA} and the Dirac Lagrangian density

$$\begin{aligned} L_D = \{ & \tfrac{i}{2}\sigma^\lambda_q [y^+_A (\gamma^0\gamma^q)^A{}_B (y^B_\lambda - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k k^\mu{}_{\lambda\nu}) L^B{}_{abC} y^C) \\ & - (y^+_{\lambda A} - \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k k^\mu{}_{\lambda\nu}) y^+_C L^{+C}{}_{abA})(\gamma^0\gamma^q)^A{}_B y^B] \\ & - m y^+_A (\gamma^0)^A{}_B y^B \}\sqrt{|\sigma|}, \qquad \sigma = \det(\sigma_{\mu\nu}) \end{aligned} \tag{8.19}$$

It is readily observed that

$$\frac{\partial L_D}{\partial k^\mu{}_{\lambda\nu}} + \frac{\partial L_D}{\partial k^\mu{}_{\nu\lambda}} = 0 \tag{8.20}$$

that is, the Dirac Lagrangian density (8.19) depends only on the torsion of a world connection.

9. GENERAL COVARIANT TRANSFORMATIONS

Since a world manifold X is parallelizable and the universal spin structure is unique, the $\widetilde{GL}_4$-principal bundle $\widetilde{LX} \to X$ as well as the frame bundle LX admit the canonical lift of any diffeomorphism f of the base X. This lift is defined by the commutative diagram

$$\begin{tikzcd} \widetilde{LX} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\tilde{z}"'] & \widetilde{LX} \arrow[d, "\tilde{z}"] \\ LX \arrow[r, "\Phi"] \arrow[d] & LX \arrow[d] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

where Φ is the holonomic bundle automorphism of LX, (2.2), induced by f (Dabrowski and Percacci, 1986).

The associated morphism of the spinor bundle S, (8.8), is given by the relation

$$\tilde{\Phi}_S\colon \quad (p, v) \cdot L_s \to (\tilde{\Phi}(p), v) \cdot L_s, \qquad p \in \widetilde{LX}, \quad v \in S \tag{9.1}$$

Because $\tilde{\Phi}$ is equivariant, this is a fiber-to-fiber automorphism of the bundle $S \to \Sigma_T$ over the canonical automorphism of the LX-associated tetrad bundle $\Sigma_T \to X$ induced by the diffeomorphism f of X. Thus, we have the commutative diagram

$$\begin{tikzcd}
S \arrow[r, "\tilde{\Phi}_S"] \arrow[d] & S \arrow[d] \\
\Sigma_T \arrow[r, "\Phi_\Sigma"] \arrow[d] & \Sigma_T \arrow[d] \\
X \arrow[r, "f"] & X
\end{tikzcd}$$

of general covariant transformations of the spinor bundle S.

Accordingly, there exists a canonical lift $\tilde{\tau}$ onto S of every vector field τ on X. The goal is to find its coordinate expression. Difficulties arise because the tetrad coordinates σ_a^μ of Σ_T depend on the choice of an atlas of the bundle $LX \to \Sigma_T$. Therefore, noncanonical vertical components appear in the coordinate expression of $\tilde{\tau}$.

A comparison with the canonical lift (2.5) of a vector field τ over the metric bundle Σ_{PR} shows that the similar canonical lift of τ over the tetrad bundle Σ_T takes the form

$$\tau_\Sigma = \tau^\lambda \partial_\lambda + \partial_\nu \tau^\mu \sigma_c^\nu \frac{\partial}{\partial \sigma_c^\mu} + Q_c^\mu \frac{\partial}{\partial \sigma_c^\mu} \tag{9.2}$$

where the terms Q_c^μ obey the condition

$$(Q_a^\mu \sigma_o^\nu + Q_a^\nu \sigma_b^\mu)\eta^{ab} = 0$$

The term $Q_a^\mu \partial_\mu^a$ is the above-mentioned noncanonical part of the lift (9.2).

Let us consider a horizontal lift $\tilde{\tau}_S$ of the vector field τ_Σ onto the spinor bundle $S \to \Sigma_T$ by means of the spin connection (8.12). It reads

$$A_S\tilde{\tau}_\Sigma = \tau^\lambda\partial_\lambda + \partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}$$
$$+ \tfrac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\sigma^\nu_k(\partial_\nu\tau^\mu - K^\mu_{\lambda\nu}\tau^\nu)(L_{ab}{}^A{}_B y^B)\,\partial_A + L^{+}_{ab}{}^A{}_B y^+_A\partial^B)$$
$$+ Q^\mu_c\frac{\partial}{\partial\sigma^\mu_c} + \frac{1}{4}Q^\mu_k(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(L_{ab}{}^A{}_B y_B\partial_A + (L^{+A}_{abB}y^+_A\partial^B)$$

Moreover, we obtain the desired canonical lift of τ onto S:

$$\tilde{\tau}_S = \tau^\lambda\partial_\lambda + \partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}$$
$$+ Q^\mu_c\frac{\partial}{\partial\sigma^\mu_c} + \frac{1}{4}Q^\mu_k(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(L_{ab}{}^A{}_B y^B\partial_A + (L^{+}_{ab}{}^A{}_B y^+_A\partial^B) \quad (9.3)$$

which can be written in the form

$$\tilde{\tau}_S = \tau^\lambda\partial^\lambda + \partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c}$$
$$\frac{1}{4}Q^\mu_k(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\left[-L^d_{abc}\sigma^\nu_d\frac{\partial}{\partial\sigma^\nu_c} + L^A_{abB}y^B\partial_A + L^{+\mathrm{A}}_{ab}y^+_A\partial^B\right]$$

where L^d_{abc} are the generators (3.4) (Giachetta *et al.,* 1997). The corresponding total vector field on the fibered product Y (8.15) reads

$$\tilde{\tau}_Y = \tilde{\tau} + \vartheta$$
$$\tilde{\tau} = \tau^\lambda\partial_\lambda + \partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c} \quad (9.4)$$
$$+ [\partial_\nu\tau^\alpha k^\nu_{\mu\beta} - \partial_\beta\tau^\nu k^\alpha_{\mu\nu} - \partial_\mu\tau^\nu k^\alpha_{\nu\beta} + \partial_{\mu\beta}\tau^\alpha]\frac{\partial}{\partial k^\alpha_{\mu\beta}}$$
$$\vartheta = \frac{1}{4}Q^\mu_k(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\left[-L^d_{abc}\sigma^\nu_d\frac{\partial}{\partial\sigma^\nu_c} + L^A_{abB}y^B\partial_A + L^{+\mathrm{A}}_{abB}y^+_A\partial^B\right]$$

Its canonical part $\tilde{\tau}$, (9.4), is the generator of a local one-parameter group of general covariant transformations of the bundle Y, whereas the vertical vector field ϑ is the generator of a local one-parameter group of principal (Lorentz) automorphisms of the bundle $S \to \Sigma_T$. By construction, the total Lagrangian density L obeys the relations

$$\mathbf{L}_{J^1\vartheta}L_D = 0 \quad (9.5)$$

$$\mathbf{L}_{J^1\tilde{\tau}}L_{MA} = 0, \qquad \mathbf{L}_{J^1\tilde{\tau}}L_D = 0 \quad (9.6)$$

The relation (9.5) leads to the Nöther conservation law, while the equalities (9.6) lead to the energy-momentum one (Giachetta and Mangarotti, 1997; Sardanashvily, 1997).

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